

Partial Differential Equations of 1st order

Partial d.E: - An equation containing 1 dependent variable and its partial derivatives w.r.t 2 or more independent variables is called a partial d.E

Ex: 1) $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 6z$

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$

4) $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z} \right)^3$

In the case of 1 dependent variable two independent variables, usually we consider z as the dependent variable & x and y are independent variables.

The first and second order partial derivatives of z with respect to x and y are $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are respectively denoted by p, q, r, s, t

i.e. $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

Order: The order of a p.d.E is defined as the order of the highest derivative occurred in the given equation.

Degree: The degree of a p.d.E is defined as the degree of the highest order derivative occurred in the given equation

Ex: $\left(\frac{\partial^2 u}{\partial x^2} \right)^3 + \frac{\partial^2 u}{\partial y^2} = 2u$ $\therefore$ is a partial d.E of order 2 and degree 3

Formation of a p.d.E:

A partial d.E can be formed by

- (i) Elimination of arbitrary constants
- (ii) " " " functions

Elimination of arbitrary constants!

* If the number of arbitrary constants to be eliminated is equal to the number of independent variables, then we get a p.d.E of 1st order

* If the number of arbitrary constants to be eliminated is greater than the number of independent variables then we get a p.d.E of 2nd and higher orders.

Sums:-

1) Form the p.d.E by eliminating the arbitrary constants a and b from $Z = ax + by + a^2 + b^2$

sol Given $Z = ax + by + a^2 + b^2$ — (1) diff. (1) partially w.r.t x & y , we get

$$\frac{\partial Z}{\partial x} = a, \quad \frac{\partial Z}{\partial y} = b$$
$$p = a \text{ — (2)} \quad q = b \text{ — (3)}$$

Sub (2) and (3) in eq (1)

$Z = px + qy + p^2 + q^2$ is the required p.d.E

2) Form the p.d.E by eliminating the arbitrary constants a and b from $2Z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

sol Given $2Z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ — (1) diff (1) partially w.r.t x & y , we get

$$2 \cdot \frac{\partial Z}{\partial x} = \frac{2x}{a^2} \Rightarrow a^2 = \frac{x}{p} \text{ — (2)}$$

$$2 \cdot \frac{\partial Z}{\partial y} = \frac{2y}{b^2} \Rightarrow b^2 = \frac{y}{q} \text{ — (3)}$$

$$\left[\begin{array}{l} \because \frac{\partial Z}{\partial x} = p \\ \frac{\partial Z}{\partial y} = q \end{array} \right]$$

Sub (2) & (3) in (1) $2Z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$

$2Z = px + qy$ is the required p.d.E

3) Find the p.d.E of all spheres whose centers lies on z-axis with radius r

Sol Equation of spheres with center (a, b, c) and radius r is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

Since center lies on z-axis $a=0, b=0$

$$x^2 + y^2 + (z-c)^2 = r^2 \quad \text{--- (1)}$$

where 'c' and 'r' are arbitrary constants

Diff Differentiating eq (1) partially w.r.t x and y

$$2x + 2(z-c) \frac{\partial z}{\partial x} = 0$$

$$(z-c) \frac{\partial z}{\partial x} = -x \Rightarrow (z-c)p = -x$$

$$z-c = -x/p \quad \text{--- (2)}$$

$$2y + 2(z-c) \frac{\partial z}{\partial y} = 0$$

$$(z-c)q = -y \Rightarrow z-c = -y/q \quad \text{--- (3)}$$

$\therefore$ from (2) and (3)

$$\frac{-x}{p} = \frac{-y}{q} \Rightarrow xq - yp = 0$$

$$3) \quad z = (-x^2 + a)(y^2 + b) \quad \text{--- (1)}$$

The no: of independent variables x, y = The no: of arbitrary constants.

Then we get first order PDE.

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differentiate eq (1) w.r.t x partially

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b)$$

$$\Rightarrow y^2 + b = \frac{p}{2x} \quad \text{--- (2)}$$

differentiate eq (1) w.r.t y partially

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a)$$

$$\Rightarrow x^2 + a = \frac{q}{2y} \quad \text{--- (3)}$$

Substitute (2) (3) in (1) we get

$$z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$\Rightarrow 4xyz = pq.$$

$$5. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\Rightarrow z^2 = c^2 \left(1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \right) \quad \text{--- (1)}$$

Here the no: of independent variables x, y = The

Here the no: of independent variables $x, y \neq$ The

no: of arbitrary constants a, b, c .

i.e The no: of arbitrary constants $>$ The no: of independent variables.

differentiate eq (1) w.r.t x partially

$$2z \frac{\partial z}{\partial x} = c^2 \left(-\frac{2x}{a^2} \right)$$

$$\Rightarrow z \frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2} \quad \text{--- (2)}$$

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differentiate w.r.t y partially

$$2z \frac{\partial z}{\partial y} = c^2 \left(-\frac{2y}{b^2} \right)$$

$$\Rightarrow z \frac{\partial z}{\partial y} = -\frac{c^2 y}{b^2} \quad \text{--- (3)}$$

again differentiate eq (2) w.r.t x partially

$$z \frac{\partial^2 z}{\partial x^2} + (1) \frac{\partial z}{\partial x} = -\frac{c^2}{a^2} \quad \text{--- (4)}$$

Substitute (4) in (2) we get

$$z \frac{\partial z}{\partial x} = \left(z \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \right) x$$

$$\Rightarrow pz = zrx + px$$

$$\Rightarrow zxr + px - px = 0$$

$$7. \quad z = xy + y\sqrt{x^2 - a^2} + b \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$p = \frac{\partial z}{\partial x} = y + y \frac{1}{2\sqrt{x^2 - a^2}} (2x) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t y partially

$$q = \frac{\partial z}{\partial y} = x + \sqrt{x^2 - a^2} \quad \text{--- (3)}$$

$$q = x + \sqrt{x^2 - a^2}$$

$$\Rightarrow \sqrt{x^2 - a^2} = q - x$$

$$\text{from (2)} \quad p = y + \frac{y}{2(q-x)} \quad \text{--- (4)}$$

$$\Rightarrow p = y + \frac{xy}{(q-x)}$$

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$$p(q-x) = y(q-x) + xy$$

$$\Rightarrow pq - px = qy - xy + xy$$

$$\Rightarrow px + qy = pq$$

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$$8. \quad (x-a)^2 + (y-b)^2 + z^2 = c^2 \quad \text{where } c \text{ is fixed.} \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$2(x-a) + 2z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) + zp = 0 \quad \text{--- (2)} \Rightarrow (x-a) = -zp$$

$$\Rightarrow (x-a) + zp = 0 \quad \text{--- (2)} \Rightarrow (x-a) = -zp$$

differentiate eq (1) w.r.t y partially

$$2(y-b) + 2z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (y-b) + zq = 0 \quad \text{--- (3)} \Rightarrow (y-b) = -zq$$

$$\text{from (1)} \quad (-zp)^2 + (-zq)^2 + z^2 = c^2$$

$$\Rightarrow z^2 p^2 + z^2 q^2 + z^2 = c^2$$

$$\Rightarrow z^2 (p^2 + q^2 + 1) = c^2$$

$$9. \quad z = axe^y + \frac{1}{2} a^2 e^{2y} + b. \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$p = \frac{\partial z}{\partial x} = ae^y \quad (1)$$

$$\Rightarrow p = ae^y \quad \text{--- (2)}$$

differentiate eq (1) w.r.t y partially

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$$q = \frac{\partial z}{\partial y} = axe^y + \frac{1}{2} a^2 e^{2y} \quad (2)$$

$$q = axe^y + a^2 e^{2y}$$

$$q = (ae^y)x + (ae^y)^2$$

$$\Rightarrow q = px + p^2 \quad [\because \text{from (2)}]$$

$$11. \log(az-1) = x+ay+b \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$\frac{1}{az-1} \left(a \frac{\partial z}{\partial x} \right) = 1$$

$$\Rightarrow ap = (az-1) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t y partially

$$\frac{1}{az-1} \cdot a \frac{\partial z}{\partial y} = a$$

$$\Rightarrow q = (az-1) \quad \text{--- (3)}$$

from (2) & (3) $ap = q$

$$\Rightarrow a = \frac{q}{p}$$

$$\text{from (3)} \quad q = \left(\frac{q}{p} \right) z - 1$$

$$\Rightarrow pq = qz - p$$

$$\Rightarrow p + pq = qz$$

$$\Rightarrow p(1+q) = qz$$

$=$

13. $z = a \log \left[\frac{b(y-1)}{1-x} \right]$

$$z = a \left[\log [b(y-1)] - \log (1-x) \right]$$

$$z = a \left[\log b + \log(y-1) - \log(1-x) \right] \text{ — (1)}$$

differentiate eq (1) w.r.t x partially

$$\frac{\partial z}{\partial x} = p = a \left[0 - \frac{1}{1-x} (-1) \right]$$

$$\Rightarrow p = a \left[\frac{1}{1-x} \right]$$

$$\Rightarrow p = \frac{a}{1-x} \text{ — (2)} \Rightarrow a = p(1-x)$$

differentiate w.r.t y partially to eq (1)

$$\frac{\partial z}{\partial y} = q = a \left[\frac{1}{(y-1)} \right]$$

$$q = \frac{a}{y-1} \text{ — (3)} \Rightarrow q(y-1) = a$$

$$(y-1)q = p(1-x)$$

$$\Rightarrow qy - q = p - px$$

$$\Rightarrow px + qy = p + q$$

14. Find the differential eq of all spheres of radius 8 and having their centres in the yz -plane.

$$x^2 + (y-b)^2 + (z-c)^2 = 64 \text{ — (1)}$$

differentiate eq (1) w.r.t x partially

$$2x + 0 + 2(z-c) \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x + (z-c)p = 0 \text{ — (2)} \Rightarrow (z-c) = \frac{-x}{p}$$

17. Find the differential eq of all spheres of radius 8 and having their centres in the yz -plane.

$$x^2 + (y-b)^2 + (z-c)^2 = 64 \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$2x + 0 + 2(z-c) \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x + (z-c)p = 0 \quad \text{--- (2)} \quad \Rightarrow (z-c) = -\frac{x}{p}$$

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differentiate eq (1) w.r.t y partially

$$2(y-b) + 2(z-c) \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (y-b) + (z-c)q = 0 \quad \text{--- (3)} \quad \Rightarrow (y-b) = -(z-c)q$$

$$= -\left(-\frac{x}{p}\right)q$$

$$(y-b) = \frac{x}{p}q$$

Substitute $(y-b)$ and $(z-c)$ in (1)

$$x^2 + \left(\frac{xq}{p}\right)^2 + \left(-\frac{x}{p}\right)^2 = 64$$

$$\Rightarrow x^2 + \frac{x^2 q^2}{p^2} + \frac{x^2}{p^2} = 64$$

$$\Rightarrow x^2 \left(1 + \frac{q^2}{p^2} + \frac{1}{p^2}\right) = 64$$

$$\Rightarrow x^2 (p^2 + q^2 + 1) = 64p^2$$

Formation of P.D.E by Eliminating arbitrary functions.

1. $z = f(x^2 + y^2)$

2. $z = f(x) + e^y g(y)$

3. $z = f\left(\frac{y}{x}\right)$

4. $z = \phi_1(x+iy) + \phi_2(x-iy)$

5. $\lambda x + my + nz = \phi(x^2 + y^2 + z^2)$

6. $\phi(x+y+z, x^2+y^2+z^2) = 0$

7. $f(x^2+y^2, z-xy) = 0$

8. $z = yf(x) + xg(y)$

9. $xyz = f(x^2 + y^2 + z^2)$

10. $z = (x+y)f(x^2-y^2)$

11. $z = e^{ax+by} f(ax-by)$

12. $z = xy + f(x^2+y^2)$

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13. $\phi(x^2+y^2+z^2, z^2-2xy) = 0$

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14. $z = f(x+ct) + g(x-ct)$

15. $z = x f(ax+by) + g(ax+by)$

16. $z = f(x-it) + g(x+it)$

17. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

18. $z = \phi_1(x) \phi_2(y)$

19. $z = x f_1(x+t) + f_2(x+t)$

20. $z = x^n f\left(\frac{y}{x}\right)$

Note Similar form of the equations

$$u = f(v) \quad (\text{or}) \quad v = f(u) \quad (\text{or}) \quad f(uv) = 0$$

2. $z = f\left(\frac{y}{x}\right)$ — (1)

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right) \quad \text{--- (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)}{f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right)}$$

$$\frac{p}{q} = -\frac{y}{x}$$

$$\Rightarrow px + qy = 0$$

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7. $f(x^2+y^2, z-xy) = 0$

$$z - xy = f(x^2+y^2) \quad (\text{or}) \quad x^2+y^2 = f(z-xy)$$

consider $x^2+y^2 = f(z-xy)$ — (1)

differentiate w.r.t 'x' partially

$$2x = f'(z-xy) \cdot \left(\frac{\partial z}{\partial x} - y\right) \quad \text{--- (2)}$$

differentiate w.r.t 'y' partially

$$2. \quad z = f(x) + e^y g(x) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = f'(x) + e^y g'(x) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = e^y g(x) \quad \text{--- (3)}$$

differentiate eq (2) w.r.t 'x' partially

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x) + e^y g''(x) \quad \text{--- (4)}$$

differentiate eq (2) w.r.t 'y' partially

$$s = \frac{\partial^2 z}{\partial x \partial y} = e^y g'(x) \quad \text{--- (5)}$$

differentiate eq (3) w.r.t 'y' partially

$$t = \frac{\partial^2 z}{\partial y^2} = e^y g(x) \quad \text{--- (6)}$$

$$\text{from (3) \& (6) } q = e^y g(x) = t$$

$$\Rightarrow q = t$$

$$\Rightarrow q - t = 0$$

$$\frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial y^2} = 0$$

=

$$2y = f'(z-xy) \left(\frac{\partial z}{\partial y} - x \right) \quad \text{--- (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2x}{2y} = \frac{f'(z-xy) (p-y)}{f'(z-xy) (q-x)}$$

$$\Rightarrow (q-x)x = (p-y)y$$

$$\Rightarrow qx - x^2 = py - y^2$$

$$\Rightarrow -py + qx = x^2 - y^2$$

=

$$9. \quad xyz = f(x^2 + y^2 + z^2) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t 'x' partially

$$y[(1)z + x \frac{\partial z}{\partial x}] = f'(x^2 + y^2 + z^2) \cdot (2x + 2z \frac{\partial z}{\partial x})$$

$$\Rightarrow y(z + xp) = f' \cdot (2x + 2zp) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$x[(1)z + y \frac{\partial z}{\partial y}] = f'(x^2 + y^2 + z^2) \cdot (2y + 2z \frac{\partial z}{\partial y})$$

$$x(z + yq) = f' \cdot (2y + 2zq) \quad \text{--- (3)}$$

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$$\frac{(2)}{(3)} \Rightarrow \frac{y(z + xp)}{x(z + yq)} = \frac{f' \cdot (2x + 2zp)}{f' \cdot (2y + 2zq)}$$

$$\Rightarrow (yz + xyp) / (y + zq) = (xz + xyp) / (y + zq)$$

$$\Rightarrow y^2z + xy^2p + yz^2q + xypq = xz^2 + x^2yq + xz^2p + xypq$$



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$$\Rightarrow px(y^2-z^2) + qy(z^2-x^2) + z(y^2-x^2) = 0.$$

$$11 \quad z = e^{ax+by} f(ax-by) \quad \text{--- (1)}$$

$$z = e^{ax} \cdot e^{by} f(ax-by) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = e^{by} \left[e^{ax} \cdot a f(ax-by) + e^{ax} f'(ax-by) \cdot a \right]$$

$$p = e^{ax+by} \cdot a \cdot f(ax-by) + a e^{ax+by} f'(ax-by)$$

$$\Rightarrow p = az + a e^{ax+by} f'(ax-by) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = e^{ax} \left[e^{by} \cdot b f(ax-by) + e^{by} f'(ax-by) (-b) \right]$$

$$q = e^{ax+by} \cdot b f(ax-by) - b e^{ax+by} f'(ax-by)$$

$$q = bz - b e^{ax+by} f'(ax-by) \quad \text{--- (3)}$$

$$\text{from (2)} \quad p - az = a e^{ax+by} f'(ax-by) \quad \text{--- (4)}$$

$$\text{and from (3)} \quad q - bz = -b e^{ax+by} f'(ax-by) \quad \text{--- (5)}$$

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$$\frac{(4)}{(5)} \Rightarrow \frac{p-az}{q-bz} = \frac{a e^{ax+by} \cdot f'}{-b e^{ax+by} \cdot f'}$$

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$$z = e^{ax} \cdot e^{by} f(ax-by) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = e^{by} \left[e^{ax} \cdot a f(ax-by) + e^{ax} f'(ax-by) \cdot a \right]$$

$$p = e^{ax+by} \cdot a \cdot f(ax-by) + a e^{ax+by} f'(ax-by)$$

$$\Rightarrow p = az + a e^{ax+by} f'(ax-by) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = e^{ax} \left[e^{by} \cdot b f(ax-by) + e^{by} f'(ax-by) (-b) \right]$$

$$q = e^{ax+by} \cdot b f(ax-by) - b e^{ax+by} f'(ax-by)$$

$$q = bz - b e^{ax+by} f'(ax-by) \quad \text{--- (3)}$$

$$\text{from (2)} \quad p - az = a e^{ax+by} f'(ax-by) \quad \text{--- (4)}$$

$$\text{and from (3)} \quad q - bz = -b e^{ax+by} f'(ax-by) \quad \text{--- (5)}$$

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$$\frac{(4)}{(5)} \Rightarrow \frac{p-az}{q-bz} = \frac{a e^{ax+by} \cdot f'}{-b e^{ax+by} \cdot f'}$$

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$$\Rightarrow -b(p-az) = a(q-bz)$$

$$\Rightarrow -bp + abz = aq - abz$$

$$\Rightarrow aq + bp = 2abz$$



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$$z = \phi_1(x+iy) + \phi_2(x-iy) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t x partially

$$p = \frac{\partial z}{\partial x} = \phi_1'(x+iy) + \phi_2'(x-iy) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t y partially

$$q = \frac{\partial z}{\partial y} = \phi_1'(x+iy)i + \phi_2'(x-iy)(-i)$$

$$q = \phi_1'(x+iy)i - \phi_2'(x-iy)i \quad \text{--- (3)}$$

differentiate eq (2) w.r.t x partially

$$r = \frac{\partial^2 z}{\partial x^2} = \phi_1''(x+iy) + \phi_2''(x-iy) \quad \text{--- (4)}$$

differentiate eq (2) w.r.t y partially

$$s = \frac{\partial^2 z}{\partial x \partial y} = \phi_1''(x+iy)i + \phi_2''(x-iy)(-i)$$

$$s = \phi_1''(x+iy)i - \phi_2''(x-iy)i \quad \text{--- (5)}$$

differentiate eq (3) w.r.t y partially

$$t = \frac{\partial^2 z}{\partial y^2} = \phi_1''(x+iy)i^2 - \phi_2''(x-iy)i^2(-i)$$

$$t = -\phi_1''(x+iy) - \phi_2''(x-iy) \quad \text{--- (6)}$$

$$= -[\phi_1''(x+iy) + \phi_2''(x-iy)]$$

$$t = -r$$

$$\Rightarrow r+t=0$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

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$$8. \quad z = y f(x) + x g(y) \quad \text{--- (1)}$$

differentiate (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = y f'(x) + g(y) \quad \text{--- (2)}$$

differentiate (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = f(x) + x g'(y) \quad \text{--- (3)}$$

differentiate (2) w.r.t 'x' partially

$$r = \frac{\partial^2 z}{\partial x^2} = y f''(x) \quad \text{--- (4)}$$

differentiate (2) w.r.t 'y' partially

$$s = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \quad \text{--- (5)}$$

differentiate (3) w.r.t 'y' partially

$$t = \frac{\partial^2 z}{\partial y^2} = x g''(y) \quad \text{--- (6)}$$

$$\text{from (2)} \quad px = xy f'(x) + x g(y)$$

$$\text{from (3)} \quad qy = y f(x) + xy g'(y)$$

$$px + qy = xy f'(x) + x g(y) + y f(x) + xy g'(y)$$

$$\Rightarrow px + qy = xy (f'(x) + g'(y)) + z$$

$$\Rightarrow px + qy = xy s + z$$

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Types of P.d.E:

There are two types of P.d.E

- 1) Linear P.d.E
- 2) Non-linear P.d.E

Linear P.d.E: A partial d.E is said to be linear if the derivatives of the dependent variable ^{appear} here in the 1st degree only and ~~do not~~ the derivatives cannot be multiplied together.

Example: $1 \cdot x^2 - \frac{\partial^2 z}{\partial x^2} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} = z$ is a linear P.d.E of order 2

2. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$ is a linear P.d.E of order 1

Non linear P.d.E: A partial d.E which is not linear, is called a non-linear P.d.E

Ex: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 0$

Solution of a P.d.E

A solution of a P.d.E is a relation b/w dependent and independent variables

Types: There are 3 types of solutions

- 1) Complete solution
- 2) Particular "
- 3) General "

Complete solution: Let $f(x, y, z, p, q) = 0$ be the P.d.E of 1st order. A relation of the form $\Phi(x, y, z, a, b) = 0$

Clearly eq. ② is a complete solution of eq. ①

($\because$ no. of arbitrary constants
= no. of independent variables)

Particular solution

A solution is obtained by giving particular values to the arbitrary constants in the complete integral is called

a particular solution

General Solution: A solution which contains arbitrary functions is called a general solution.

Ex:- $z = f(x+y)$ is a general solution of $p-q=0$ where 'f' is arbitrary function.

Linear Partial d.E

Lagrange's Linear equation:

A first order linear P.D.E is of the form

$Pp + Qq = R$ where P, Q, R are functions of x, y, z is called a Lagrange's linear equation.

Let $f(u, v) = 0$ be the general solution of eq-①

If ① is obtained from ② by eliminating the arbitrary function 'f' where u and v are independent functions of x, y, z

Working Rule to solve $Pp + Qq = R$: — ①

Step-1: write the Lagrange's Auxiliary equations (subsidiary equations)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- ②}$$

Step-2: Find any two independent solutions of subsidiary equations

Let the solutions be $u = a, v = b$

Step-3: The general solution of eq-① is given by $f(u, v) = 0$

Method of grouping:-

In some problems, It is possible that the

subsidiary equations are directly solvable to get the solutions. Some times the subsidiary equations are not convenient to solve using method of grouping. In such cases we choose the method

of multipliers.

Each ratio is equal to the $\frac{l dx + m dy + n dz}{l p + m q + n r}$

where l, m, n are called as Multipliers

If l, m, n are so chosen that $l p + m q + n r = 0$ then

$$l dx + m dy + n dz = 0$$

Integrating this, we get one independent solution (say $u = a$)

Similarly we get another independent solution (say $v = b$)

Sums:

1) Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

sol) Given $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ — ①

eq-① is of the form $Pp + Qq = R$

Here $P = \sqrt{x}, Q = \sqrt{y}, R = \sqrt{z}$

The lagranger Auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Taking 1st two Ratios

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

I.O.B.S

$$\int x^{-1/2} dx = \int y^{-1/2} dy$$

$$2\sqrt{x} = 2\sqrt{y} + C$$

$$\sqrt{x} - \sqrt{y} = C/2 \Rightarrow \boxed{\sqrt{x} - \sqrt{y} = K_1}$$

Taking last two ratios $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$ I.O.B.S

$$\int y^{-1/2} dy = \int z^{-1/2} dz$$

$$2\sqrt{y} = 2\sqrt{z} + C_1$$

$$\sqrt{y} - \sqrt{z} = C_1/2 \quad \therefore$$

$$\sqrt{y} \cdot \sqrt{z} = K_2 \quad (\because K_2 = C_1/2)$$

$\therefore$ General solution is $f(\sqrt{z} - \sqrt{y}, \sqrt{y} \cdot \sqrt{z}) = 0$

Q) solve $\tan x \cdot p + \tan y \cdot q = \tan z$

sol] Given $\tan x \cdot p + \tan y \cdot q = \tan z$ — (1)

It is of the form $P = Q, R$

$$P = \tan x, Q = \tan y, R = \tan z$$

The Lagrange's Auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking 1st two equations

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow I \cdot O \cdot B \cdot S$$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos y}{\sin y} dy$$

$$\log(\sin x) = \log(\sin y) + \log C$$

$$\frac{\sin x}{\sin y} = K_1$$

Taking last two

$$\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow I \cdot O \cdot B \cdot S \Rightarrow \int \frac{\cos y}{\sin y} dy = \int \frac{\cos z}{\sin z} dz$$

$$\log(\sin y) = \log(\sin z) + \log C$$

$$\frac{\sin y}{\sin z} = K_2$$

$\therefore$ The g.s is $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

3) Solve $\frac{y^2 z}{x} \cdot P + x^2 z q = y^2$

$$y^2 z \cdot P + x^2 z q = y^2 \quad \text{--- (1)}$$

eq-① is of the form $Pp + Qq = R$

$$P = y^2 z, Q = x^2 z, R = y^2$$

The Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Comparing 1st two co-efficients

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} \Rightarrow x^2 dx = y^2 dy$$

I.O.B.S

$$\int x^2 dx = \int y^2 dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + C$$

$$\frac{x^3}{3} - \frac{y^3}{3} = C = K_1$$

$$\frac{dx}{y^2 z} = \frac{dz}{xy^2}$$

$$x dx = z dz \quad \text{I.O.B.S}$$

$$\frac{x^2}{2} - \frac{z^2}{2} = C \Rightarrow x^2 - z^2 = K_2$$

$\therefore$ The g.s is $f(x^3 - y^3, x^2 - z^2) = 0$.

4) Solve $x^2(y-z)P + y^2(z-x)Q = z^2(x-y)$

sol $x^2(y-z)P + y^2(z-x)Q = z^2(x-y)$

eq-① is of the form $Pp + Qq = R$

$$P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$$

The Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

here the auxiliary equations are not convenient to solve by the method of grouping. so we choose Method of Multipliers

Taking $l = 1/x, m = 1/y, n = 1/z$ as multipliers are so chosen that
 $lp + mq + nR = 0$ then $l dx + m dy + n dz = 0$

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

I.O.B.s

$$\int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = 0$$

$$\log x + \log y + \log z = \log c$$

$$\boxed{xyz = c}$$

Again let $l = 1/x^2, m = 1/y^2, n = 1/z^2$ such that

$$lp + mq + nR = 0 \text{ then } l dx + m dy + n dz = 0$$

$$\frac{1}{x^2} dx + \frac{1}{y^2} dy + \frac{1}{z^2} dz = 0$$

I.O.B.s

$$\int \frac{1}{x^2} dx + \int \frac{1}{y^2} dy + \int \frac{1}{z^2} dz = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = K \Rightarrow \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -K}$$

$\therefore$ The general solution of eq. (1) is

$$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

5) Solve $xp - yq = y^2 - x^2$

sol Given $xp - yq = y^2 - x^2$ is of the form $Pp + Qq = R$

$$\text{Here } P = x, Q = -y, R = y^2 - x^2$$

The Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

from the first two terms $\frac{dx}{x} = \frac{dy}{-y}$

I.O.B.s

$$\int \frac{1}{x} dx = -\int \frac{1}{y} dy \Rightarrow \log x = -\log y + \log c$$

$$\log(xy) = \log c$$

$$\boxed{xy = c}$$

let $l=x, m=y, n=1$ so chosen that

$$lp + mq + nr = 0$$

$$\therefore ldx + mdy + ndz = 0$$

$$x dx + y dy + dz = 0$$

I.O.B.s

$$\int x dx + \int y dy + \int dz = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + z = K$$

$$\Rightarrow \boxed{x^2 + y^2 + 2z = 2K}$$

The general solution of given equation is $f(xy, x^2 + y^2 + 2z) = 0$

6) Solve $(y^2 + z^2)p - xyq + zx = 0$

sol) Given $(y^2 + z^2)p - xyq = -zx$

It is of the form $Pp + Qq = R$

$$P = y^2 + z^2, Q = -xy, R = -zx$$

$\therefore$ The Auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

$\therefore$ from last two $\frac{dy}{-xy} = \frac{dz}{-zx}$

I.O.B.s

$$\int \frac{1}{y} dy = \int \frac{1}{z} dz \Rightarrow \log y = \log z + \log c$$

$$y = zc \Rightarrow \boxed{y/z = c}$$

Taking $l=x, m=y, n=z$ are so chosen that $lp + mq + nr = 0$

$$l dx + m dy + n dz = 0 \text{ and } 1$$

$$x dx + y dy + z dz = 0 \text{ I.O.B.s}$$

$$\int x dx + \int y dy + \int z dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = K \Rightarrow \sqrt{x^2 + y^2 + z^2} = \sqrt{2K}$$

∴ The given auxiliary equation is $f(y/z, x^2 + y^2 + z^2) = 0$

7) Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

sol Given $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

It is of the form $Pp + Qq = R$

$$P = x^2 - y^2 - z^2, Q = 2xy, R = 2xz$$

$$\therefore \frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{--- I}$$

∴ from last two $\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \text{I.O.B.S}$

$$\int \frac{1}{y} dy = \int \frac{1}{z} dz \Rightarrow \log y = \log z + \log x$$

$$\Rightarrow \boxed{y/z = C}$$

let $l = x, m = y, n = z$ and each ratio is equal to $\frac{l dx + m dy + n dz}{l p + m q + n R}$

$$\frac{l dx + m dy + n dz}{x^2 - y^2 - z^2 + 2xy + 2xz}$$

$$= \frac{x^3 - y^3 - z^3 + 2xy^2 + 2xz^2}{2}$$

$$= \frac{2(x^2 + y^2 + z^2)}{2}$$

Each ratio = $\frac{x dx + y dy + z dz}{x^2 + y^2 + z^2} = \frac{dz}{2xz}$

$$\frac{2x dx + 2y dy + 2z dz}{x^2 + y^2 + z^2} = \frac{1}{2} dz$$

I.O.B.S

$$\log(x^2 + y^2 + z^2) = \log z + \log K$$

$$x^2 + y^2 + z^2 = zK \Rightarrow \frac{x^2 + y^2 + z^2}{z} = K$$

∴ General solution is $F(y/z, \frac{x^2 + y^2 + z^2}{z}) = 0$

Non Linear Partial d.E:-

There are four types of non-linear p.d.E's.

1) $f(p, q) = 0$

2) $f(z, p, q) = 0$

3) $f(x, p) = g(y, q)$

4) $z = px + qy + f(p, q)$

Standard form-1:-

Equations of the form $f(p, q) = 0$ — (1)

i.e., equations containing p and q only

Working Rule:-

Let $z = ax + by + c$ be a ^{trial} solution of equation (1) — (2)

then $\frac{\partial z}{\partial x} = a$ i.e., $p = a$

similarly $\frac{\partial z}{\partial y} = b$ i.e., $q = b$

Substituting these values in eq-①, we get

$$f(a, b) = 0$$

To find the complete solution of eq-①, we have to eliminate any one of the constant from (2)

$\therefore$ from (2), we can express b in terms of a

let $b = \phi(a)$

Then the required complete solution is

$$\boxed{z = ax + \phi(a) \cdot y + c}$$

Sums:-

1) Solve $p^2 + q^2 = 1$

Sol Given $p^2 + q^2 = 1$ — (1)

eq-① is of the form $f(p, q) = 0$

let $z = ax + by + c$ be a trial solution of eq-① — (2)

$$\frac{\partial z}{\partial x} = a, \text{ i.e., } p = a \text{ and } \frac{\partial z}{\partial y} = b \text{ i.e., } q = b$$

Substituting these values in (1), we get

$$a^2 + b^2 = 1 \quad \text{--- (3)}$$

To find the complete solution, we have to eliminate any one of the constant from (2)

$$\therefore \text{ from (3) } b^2 = 1 - a^2 \Rightarrow b = \pm \sqrt{1 - a^2}$$

Sub 'b' value in (2), we get

$$z = ax \pm \sqrt{1 - a^2} y + c \text{ is the required complete solution.}$$

2) Solve $pq + p + q = 0$

Sol Given $pq + p + q = 0$ --- (1)

It is of the form $f(p, q) = 0$

let $z = ax + by + c$ is the trial solution of eq. (1)
--- (2)

$$\frac{\partial z}{\partial x} = a \text{ i.e., } p = a, \quad \frac{\partial z}{\partial y} = b \text{ i.e., } q = b$$

Sub these values in (1), we get

$$ab + a + b = 0 \quad \text{--- (3)}$$

To find the complete solution we have to eliminate any one of the constant from (2)

$$\therefore \text{ from (3) } b(a+1) + a = 0 \Rightarrow b = -\frac{a}{a+1}$$

$\therefore$ Sub 'b' value in eq. (2), we get

$$z = ax - \frac{a}{a+1} y + c \text{ is the required complete solution.}$$

(3) Solve $p^2 + q^2 = npq$

Given $p^2 + q^2 = npq$ — (1)

eq (1) is of the form standard form — (1)

let $z = ax + by + c$ be the trial solution — (2)

Then $\frac{\partial z}{\partial x} = a \Rightarrow p = a$

$$\frac{\partial z}{\partial y} = b(1) \Rightarrow q = b$$

Sub these values in (1)

$$a^2 + b^2 = nab$$

$$a^2 - nab + b^2 = 0$$

$$\therefore a = \frac{nb \pm \sqrt{n^2 b^2 - 4b^2}}{2}$$

$$= \frac{b}{2} [n \pm \sqrt{n^2 - 4}] \text{ — (3)}$$

sub (3) in (2), we get

$$z = bx + \frac{b}{2} y [n \pm \sqrt{n^2 - 4}] + c$$

Equations reducible to standard forms

Equations of the form $f(x^m p, y^n q) = 0$ — (1)

eq-① can be reduced to an equation of the form

$$f(P, Q) = 0$$

By the following transformations

Case (i):- When $m=1, n=1$ then the eq-① becomes,

$$f(px, qy) = 0.$$

For the combination px , we use the transformation

$$x = \log x$$

For the combination qy , we use the transformation

$$y = \log y$$

W.K.T

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} = \frac{\partial z}{\partial x} \left(\frac{1}{x} \right)$$

$$px = \frac{\partial z}{\partial x} = p \text{ (say)}$$

$$qy = \frac{\partial z}{\partial y} = Q \text{ (say)}$$

Now eq-① will be reduced to the form

$$f(p, Q) = 0$$

Case (ii):- When $m \neq 1$ and $n \neq 1$

$$\text{Let } x = x^{1-m}, y = y^{1-n}$$

$$\text{then } \frac{\partial x}{\partial x} = (1-m)x^{-m} \text{ and } \frac{\partial y}{\partial y} = (1-n)y^{-n}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} = \frac{\partial z}{\partial x} (1-m)x^{-m}$$

$$\Rightarrow x^m p = (1-m) \frac{\partial z}{\partial x} = (1-m)p$$

$$\text{Similarly } y^n q = (1-n) \frac{\partial z}{\partial y} = (1-n)Q$$

Now eq-① will be transformed to $f((1-m)p, (1-n)Q) = 0$

Sums:-

1) Solve $x^2 p^2 + y^2 q^2 = 1$

Sol Given equation can be written as $(px)^2 + (qy)^2 = 1$ ①

eq-① will be transformed in the form of $f(P, Q) = 0$

By $x = \log x$ & $y = \log y$

$$\frac{\partial x}{\partial x} = \frac{1}{x} \quad \frac{\partial y}{\partial y} = \frac{1}{y}$$

$$\text{W.K.T } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{1}{x}$$

$$\Rightarrow px = \frac{\partial z}{\partial x} = P$$

$$11'ly \quad qy = \frac{\partial z}{\partial y} = Q$$

Sub these values in ①, we get

$$p^2 + Q^2 = 1 \quad \text{--- ②}$$

let $z = ax + by + c$ be a solution of ②

$$\text{then } \frac{\partial z}{\partial x} = a \quad \left| \quad \frac{\partial z}{\partial y} = b \right. \\ p = a \quad \left| \quad Q = b \right.$$

$$\therefore \text{from ② } a^2 + b^2 = 1 \Rightarrow a^2 = 1 - b^2 \Rightarrow b = \pm \sqrt{1 - a^2}$$

Sub 'b' values in ③, we get

$$z = ax \pm \sqrt{1 - a^2}y + c$$

$$\boxed{z = a \log x \pm \sqrt{1 - a^2} \log y + c} \text{ is the r.c.s}$$

Standard form-2:

equations of the form $f(z, p, q) = 0$ i.e. equations not containing x and y . --- ①

Working Rule:

Let $z = f(u)$ where $u = x + ay$ be a tentative solution

$$\text{of eq-①. Then } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} (1) = \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial z}{\partial u} (a) = a \cdot \frac{\partial z}{\partial u}$$

Substituting these values in eq-①, we get

$$f\left(z, \frac{\partial z}{\partial u}, a \cdot \frac{\partial z}{\partial u}\right) = 0 \quad \text{or } f\left(z, \frac{dz}{du}, a \cdot \frac{dz}{du}\right) = 0$$

which is an ordinary d.E of 1st order

The solution of above d.E will give the complete solution

of eq-①

Sums

$$1) \text{ Solve } z = p^2 + q^2$$

$$\text{sol Given } z = p^2 + q^2 \quad \text{--- ①}$$

eq-① is of the form $f(z, p, q) = 0$

let $z = f(u) \Rightarrow u = x + ay$ be a trial solution of eq-①

$$\text{then } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} (1) = \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \frac{\partial z}{\partial u}$$

Sub these values in ①, we get

$$z = \left(\frac{\partial z}{\partial u}\right)^2 + \left(a \frac{\partial z}{\partial u}\right)^2 \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = z$$

$$\Rightarrow z = \left(\frac{dz}{du}\right)^2 (1+a^2)$$

$$\frac{z}{1+a^2} = \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}} \Rightarrow \int z^{-1/2} dz = \frac{1}{\sqrt{1+a^2}} \int du + c$$

$$\frac{z^{1/2}}{1/2} = \frac{1}{\sqrt{1+a^2}} (u) + c$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$$

2) solve $q^2 = z^2 p^2 (1-p^2)$

sol] Given $q^2 = z^2 p^2 (1-p^2)$ — ①

eq-① is of the form $f(z, p, q) = 0$

let $z = f(u) \Rightarrow u = ay + x$ be a trial solution of eq-①

$$\text{then } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} (1)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \frac{\partial z}{\partial u}$$

sub these values in ①, we get

$$\left(a \frac{\partial z}{\partial u}\right)^2 = z^2 \left(\frac{\partial z}{\partial u}\right)^2 \left(1 - \left(\frac{\partial z}{\partial u}\right)^2\right)$$

$$a^2 \left(\frac{dz}{du}\right)^2 = z^2 \left(\frac{dz}{du}\right)^2 \left(1 - \left(\frac{dz}{du}\right)^2\right)$$

$$1 - \left(\frac{dz}{du}\right)^2 = \frac{a^2}{z^2}$$

$$1 - \frac{a^2}{z^2} = \left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \sqrt{\frac{z^2 - a^2}{z^2}} = \frac{\pm \sqrt{z^2 - a^2}}{z}$$

$$\frac{dz}{du} = \frac{z dz}{\sqrt{z^2 - a^2}} = \pm du$$

I.O.B.S

$$\frac{1}{2} \int \frac{2z dz}{\sqrt{z^2 - a^2}} = \pm \int du$$

$$\frac{1}{2} \cdot 2 \sqrt{z^2 - a^2} = \pm u + c$$

$$\sqrt{z^2 - a^2} = (u + c)^\pm = (x + ay + c)^\pm \text{ is the required r.s.}$$

3) Solve $x^2 p^2 + y^2 q^2 = z^2$

sol Given equation can be written as $(px)^2 + (qy)^2 = z^2$ — (1)

Let $x = \log x$, $y = \log y$ then

$$\frac{\partial x}{\partial x} = \frac{1}{x}, \quad \frac{\partial y}{\partial y} = \frac{1}{y}$$

$$\text{W.K.T } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{1}{x}$$

$$px = \frac{\partial z}{\partial x} \text{ and similarly } qy = \frac{\partial z}{\partial y}$$

Sub these values in (1), we have

$$\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = z^2$$

$$p^2 + q^2 = z^2 \text{ — (2)}$$

(2) is of form $f(p, q, z) = 0$

Let $z = f(u)$, where $u = x + ay$ then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} (1)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial z}{\partial u} (a)$$

Sub p, q values in (2), we get

$$\left(\frac{\partial z}{\partial u} \right)^2 + (a \frac{\partial z}{\partial u})^2 = z^2 \text{ (or) } \left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 = z^2$$

$$\left(\frac{dz}{du} \right)^2 [1 + a^2] = z^2$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z} dz = \frac{1}{\sqrt{1+a^2}} \cdot du$$

I.O.B.S

$$\int \frac{1}{z} \cdot dz = \frac{1}{\sqrt{1+a^2}} \int 1 \cdot du + C$$

$$\log z = \frac{u}{\sqrt{1+a^2}} + C \Rightarrow \log z = \frac{x+ay}{\sqrt{1+a^2}} + C$$

$$\boxed{\log z = \frac{\log x + a \log y}{\sqrt{1+a^2}} + C} \text{ is the required complete solution}$$

Standard form-3

Equations of the form $f(x, p) = g(y, q)$ i.e. equations not containing z and the equations containing x and p can be separated from those containing y and q .

Working Rule :

* Consider $f(x, p) = g(y, q)$ — (1) As a trial solution, we assume that each term of eq-① is equal to an arbitrary constant 'a'.

$$\text{i.e., } f(x, p) = g(y, q) = a$$

Solving for p and q we have

$$p = F_1(x, a), \quad q = F_2(y, a)$$

$\therefore z$ is a function of x and y then the total derivative of z

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy \quad \text{--- (2)}$$

Substituting the values of p, q in eq-②, we have

$$dz = F_1(x, a) dx + F_2(y, a) dy$$

$$\text{I.O.B.S} \Rightarrow \int dz = \int F_1(x, a) dx + \int F_2(y, a) dy$$

$$\therefore \boxed{z = \int F_1(x, a) dx + \int F_2(y, a) dy} \text{ is the r.e.s.}$$

Sums:

$$1) \text{ Solve } yp + xq + pq = 0$$

$$\text{sol} \text{ Given } yp + xq + pq = 0 \quad \text{--- (1)}$$

$$\text{Dividing (1) with } pq \Rightarrow \frac{y}{q} + \frac{x}{p} + 1 = 0$$

$$\frac{x}{p} = -\left(1 + \frac{y}{q}\right)$$

Assume that each term is equal to an arbitrary constant

$$\frac{x}{p} = -(1 + \frac{y}{q}) = a$$

$$\frac{x}{p} = a \Rightarrow p = \frac{x}{a} \parallel -(1 + \frac{y}{q}) = a \Rightarrow -\frac{y}{q} = 1 + a \Rightarrow q = -\frac{y}{1+a}$$

Sub p, q values in $dz = p dx + q dy$

$$dz = \frac{x}{a} dx - \frac{y}{1+a} dy \Rightarrow I.O.B.s$$

$$\int dz = \frac{1}{a} \int x dx - \frac{1}{1+a} \int y dy + C$$

$$z = \frac{x^2}{2a} - \frac{y^2}{2(1+a)} + C \text{ is the r.c.s}$$

2) Solve $p^2 + q^2 = x^2 + y^2$

sol Given equation can be written as $p^2 - x^2 = y^2 - q^2$ — (1)

Assume that each term is equal to arbitrary Constant

$$p^2 - x^2 = y^2 - q^2 = a^2 \text{ (say)}$$

$$p^2 - x^2 = a^2 \parallel y^2 - q^2 = a^2$$

$$p = \sqrt{x^2 + a^2} \parallel \Rightarrow q = \sqrt{y^2 - a^2}$$

Sub these values in $dz = p dx + q dy$

$$dz = \sqrt{a^2 + x^2} dx + \sqrt{y^2 - a^2} dy \Rightarrow I.O.B.s$$

$$\int dz = \int \sqrt{a^2 + x^2} dx + \int \sqrt{y^2 - a^2} dy + C$$

$$z = \frac{x}{a} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1}(\frac{x}{a}) + \frac{y}{a} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}(\frac{y}{a}) + C$$

3) Solve $\sqrt{p} + \sqrt{q} = x^2$

Given equation can be written as $\sqrt{p} - x^2 = -\sqrt{q} = a$

Solving for p, q

$$\sqrt{p} - x^2 = a$$

$$\sqrt{p} = a + x^2$$

$$p = (a + x^2)^2$$

$$-\sqrt{q} = a$$

$$q = a^2$$

Sub p, q values in $dz = p dx + q dy$

$$= (a + x^2)^2 dx + a^2 dy$$

$$\int dz = \int (a + x^2)^2 dx + a^2 \int dy + C$$

Sub these values in $dz = p dx + q dy$

$$dz = (\sqrt{a} - x) dx + (\sqrt{1-a} - y) dy$$

$$I.O.B.S$$

$$\int dz = \int (\sqrt{a} - x) dx + \int (\sqrt{1-a} - y) dy + c$$

$$Z = \sqrt{a}x - \frac{x^2}{2} + \sqrt{1-a}y - \frac{y^2}{2} + c$$

$$\frac{z^2}{2} = \sqrt{a}x + \sqrt{1-a}y - \left(\frac{x^2+y^2}{2}\right) + c \text{ is the r.c.s}$$

Standard form-IV:-

Equations of the form $z = px + qy + f(p, q)$

This equation is known as Clairaut's equation

Consider $z = px + qy + f(p, q)$ — ①

A relation of the form $z = ax + by + f(a, b)$ — ②

differentiating eq-② partially w.r.t x and y

$$\frac{\partial z}{\partial x} = a \quad \frac{\partial z}{\partial y} = b$$

$$p = a \quad q = b$$

eliminating a, b from ②, we get eq-①

$\therefore z = ax + by + f(a, b)$ is the complete solution of eq-①

Sums:-

1) Solve $z = px + qy + 2\sqrt{pq}$

Given $z = px + qy + 2\sqrt{pq}$ — ①

eq-① is of the form $px + qy + f(p, q) = z$

The complete solution of eq-① is obtained by writing $p = a$ and

$q = b$

$$\therefore z = ax + by + 2\sqrt{ab}$$

2) Solve $(p+q)(z - px - qy) = 1$

Given equation can be written as

$$z - px - qy = \frac{1}{p+q}$$

$$z = px + qy + \frac{1}{p+q} \text{ — ①}$$

equation ① is of the form $px + qy + f(p, q) = z$

The complete solution of eq-① can be obtained by writing $p = a, q = b$

$$\therefore z = ax + by + \frac{1}{a+b}$$

3) solve $pqz = p^2(qx + p^2) + q^2(py + q^2)$ Given $pqz = p^2(qx + p^2) + q^2(py + q^2)$

sol] Given equation can be written as

Dividing ① with pq

$$z = px + \frac{p^3}{q} + qy + \frac{q^3}{p} \quad \text{--- ①}$$

$$= px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$

eq-① is of the form $z = px + qy + f(p, q)$

The complete solution of eq-① can be obtained by writing $p = a$ and $q = b$

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

4) solve $z = px + qy + x \log x \cdot p + y \log y \cdot q$

sol] Given equation can be written as $z = px + qy + \log x (px) + \log y (qy)$ --- ①

Eq-① can be reduced to the standard form ④ by the transformation $X = \log x$ and $Y = \log y$

$$\frac{\partial x}{\partial X} = \frac{1}{x}, \quad \frac{\partial y}{\partial Y} = \frac{1}{y}$$

$$\text{W.K.T } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \left(\frac{1}{x} \right)$$

$$\therefore px = \frac{\partial z}{\partial X} = P$$

$$\text{Similarly } qy = \frac{\partial z}{\partial Y} = Q$$

Sub these values in ①, we get

$$z = P + Q + XP + YQ$$

$$z = PX + QY + P + Q \quad \text{--- ②}$$

eq-② is of the form standard form ④

The complete solution of the eq-② is obtained by replacing

$$p = a, q = b$$

$$Z = ax + by + a + b$$

$$= a \log x + b \log y + a + b$$

1) Solve $\frac{x^r}{p} + \frac{y^r}{q} = z$

Sol:- Given equation can be written as $(x^r p)^{-1} + (y^r q)^{-1} = z$ — ①

This is of the form $f(x^m p, y^n q, z) = 0$ with $m = -2$ and $n = -2$

$\therefore$ put $X = x^{1-m} = x^{1-(-2)} = x^3$ and $Y = y^{1-n} = y^3$.

Now $p = \frac{dz}{dx} = \frac{dz}{dX} \frac{dX}{dx} = p \cdot 3x^2$ where $p = \frac{dz}{dX} \Rightarrow x^2 p = 3P$

|| $y^2 q = 3Q$ where $Q = \frac{dz}{dY}$.

Now ① becomes $(3P)^{-1} + (3Q)^{-1} = z$

This is of the form $f(z, P, Q) = 0$ — ②

Let $z = g(X + aY)$ be the solution of ②

put $u = X + aY$ then $z = g(u)$

$\therefore p = \frac{dz}{du}$ and $Q = a \frac{dz}{du}$

Now ② becomes $(3 \frac{dz}{du})^{-1} + (3a \frac{dz}{du})^{-1} = z$

i.e. $\frac{1}{3} \frac{du}{dz} + \frac{1}{3a} \frac{du}{dz} = z$.

(or) $\frac{1}{3} \frac{du}{dz} [1 + \frac{1}{a}] = z$, Separating the variables,

$z dz = (\frac{a+1}{3a}) du$, Integrating, we get

$$\frac{z^2}{2} = (\frac{a+1}{3a}) u + c$$

$$\Rightarrow \frac{z^2}{2} = \frac{a+1}{3a} (x + ay) + c$$

$$\Rightarrow \frac{z^2}{2} = (\frac{a+1}{3a}) [x^3 + ay^3] + c, \text{ is the required solution,}$$

②. Solve $z^r(p^r + q^r) = x^r + y^r$

Sol:- Given equation can be written as $(pz)^r - x^r = y^r - (qz)^r$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$

∴ put $Z = z^{n+1} = z^{1+1} = z^2$ then

$$\frac{dZ}{dx} = 2z \frac{dz}{dx} \Rightarrow p = 2zp \text{ (or) } pz = \frac{p}{2} \text{ and } \frac{dZ}{dy} = 2z \frac{dz}{dy}$$

$$\Rightarrow Q = 2zq \text{ (or) } qz = \frac{Q}{2}$$

Substituting the values of pz and qz in (1), we have

$$\left(\frac{p}{2}\right)^r - x^r = y^r - \left(\frac{Q}{2}\right)^r$$

$$\text{i.e. } p^r - 4x^r = 4y^r - Q^r$$

which is of the form $f_1(x, p) = f_2(y, Q)$

$$\text{i.e. } p^r - 4x^r = 4y^r - Q^r = 4a^r \text{ (say)}$$

$$\therefore p = \sqrt{4a^r + 4x^r} = 2\sqrt{x^r + a^r}$$

$$Q = \sqrt{4y^r - 4a^r} = 2\sqrt{y^r - a^r}$$

$$\text{d } dZ = p dx + Q dy = 2\sqrt{x^r + a^r} dx + 2\sqrt{y^r - a^r} dy$$

I.O.B.S.

$$Z = 2 \int \sqrt{x^r + a^r} dx + 2 \int \sqrt{y^r - a^r} dy$$

$$= 2 \left[\frac{x}{2} \sqrt{x^r + a^r} + \frac{a^r}{2} \sinh^{-1} \left(\frac{x}{a} \right) \right] + 2 \left[\frac{y}{2} \sqrt{y^r - a^r} - \frac{a^r}{2} \cosh^{-1} \left(\frac{y}{a} \right) \right] + C$$

$$\text{(or) } z^r = x \sqrt{x^r + a^r} + a^r \sinh^{-1} \left(\frac{x}{a} \right) + y \sqrt{y^r - a^r} - a^r \cosh^{-1} \left(\frac{y}{a} \right) + C$$

$$= x \sqrt{x^r + a^r} + y \sqrt{y^r - a^r} + a^r \left[\sinh^{-1} \left(\frac{x}{a} \right) - \cosh^{-1} \left(\frac{y}{a} \right) \right] + C$$